2023

AP®

AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 1

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

Part A (AB or BC): Graphing calculator required

Question 1 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table.

Model Solution	Scoring	

(a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals $[60, 90]$, $[90, 120]$, and $[120, 135]$ to approximate the value of $\int_{60}^{135} f(t)\,dt$.

Model Solution	Scoring	
$\int_{60}^{135} f(t)\,dt$ represents the total number of gallons of gasoline pumped into the gas tank from time $t = 60$ seconds to time $t = 135$ seconds.	Interpretation with units	**1 point**
$\int_{60}^{135} f(t)\,dt$ $\approx f(90)(90-60) + f(120)(120-90) + f(135)(135-120)$ $= (0.15)(30) + (0.1)(30) + (0.05)(15) = 8.25$	Form of Riemann sum	**1 point**
	Answer	**1 point**

Scoring notes:

- To earn the first point the response must reference gallons of gasoline added/pumped and the time interval $t = 60$ to $t = 135$.
- To earn the second point at least five of the six factors in the Riemann sum must be correct.
- If there is any error in the Riemann sum, the response does not earn the third point.
- A response of $(0.15)(30) + (0.1)(30) + (0.05)(15)$ earns both the second and third points, unless there is a subsequent error in simplification, in which case the response would earn only the second point.

- A response that presents a correct value with accompanying work that shows the three products in the Riemann sum but does not show all six of the factors and/or the sum process, does not earn the second point but does earn the third point. For example, responses of either $4.5 + 3.0 + 0.75$ or $(0.15)(30),\ 0.1(30),\ 0.05(15) \rightarrow 8.25$ earn the third point but not the second.
- A response of $f(90)(90-60) + f(120)(120-90) + f(135)(135-120) = 8.25$ earns both the second and the third points.
- A response that presents an answer of only 8.25 does not earn either the second or third point.
- A response that provides a completely correct left Riemann sum with accompanying work, $f(60)(30) + f(90)(30) + f(120)(15) = 9$, or $(0.1)(30) + (0.15)(30) + (0.1)(15)$ earns 1 of the last 2 points. A response with any errors or missing factors in a left Riemann sum earns neither of the last 2 points.

Total for part (a) 3 points

(b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.

f is differentiable. $\Rightarrow$ f is continuous on $[60, 120]$.	$f(120) - f(60) = 0$	**1 point**
$\dfrac{f(120) - f(60)}{120 - 60} = \dfrac{0.1 - 0.1}{60} = 0$	Answer with justification	**1 point**
By the Mean Value Theorem, there must exist a c, for $60 < c < 120$, such that $f'(c) = 0$.		

Scoring notes:

- To earn the first point a response must present either $f(120) - f(60) = 0$, $0.1 - 0.1 = 0$ (perhaps as the numerator of a quotient), or $f(60) = f(120)$.
- To earn the second point a response must:
  - o have earned the first point,
  - o state that f is continuous because f is differentiable (or equivalent), and
  - o answer "yes" in some way.
- A response may reference either the Mean Value Theorem or Rolle's Theorem.
- A response that references the Intermediate Value Theorem cannot earn the second point.

Total for part (b) 2 points

(c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$. Show the setup for your calculations.

$\dfrac{1}{150 - 0}\displaystyle\int_0^{150} g(t)\, dt$	Average value formula	**1 point**
$= 0.0959967$	Answer	**1 point**

The average rate of flow of gasoline, in gallons per second, is 0.096 (or 0.095).	

Scoring notes:

- The exact value of $\frac{1}{150}\int_0^{150} g(t)\,dt$ is $\frac{12}{125}\sin\left(\frac{25}{16}\right)$.
- A response may present the average value formula in single or multiple steps. For example, the following response earns both points: $\int_0^{150} g(t)\,dt = 14.399504$ so the average rate is 0.0959967.
- A response that presents the average value formula in multiple steps but provides incorrect or incomplete communication (e.g., $\int_0^{150} g(t)\,dt = \frac{14.399504}{150} = 0.0959967$) earns 1 out of 2 points.
- A response of $\int_0^{150} g(t)\,dt = 0.0959967$ does not earn either point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, $\frac{1}{150}\int_0^{150} g(t)\,dt = 0.149981$ or 0.002618.

Total for part (c) 2 points

(d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

$g'(140) \approx -0.004908$ $g'(140) = -0.005$ (or -0.004)	$g'(140)$	**1 point**
The rate at which gasoline is flowing into the tank is decreasing at a rate of 0.005 (or 0.004) gallon per second per second at time $t = 140$ seconds.	Interpretation	**1 point**

Scoring notes:

- The exact value of $g'(140)$ is $\frac{1}{500}\cos\left(\frac{49}{36}\right) - \frac{49}{9000}\sin\left(\frac{49}{36}\right)$.
- The value of $g'(140)$ may appear only in the interpretation.
- To be eligible for the second point a response must present some numerical value for $g'(140)$.
- To earn the second point the interpretation must include "the rate of flow of gasoline is changing at a rate of [the declared value of $g'(140)$]" and "at $t = 140$" (or equivalent).
- An interpretation of "decreasing at a rate of -0.005" or "increasing at a rate of 0.005" does not earn the second point.
- Degree mode: In degree mode, $g'(140) = 0.001997$ or 0.00187.

Total for part (d) 2 points

Total for question 1 9 points

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Answer QUESTION 1 parts (a) and (b) on this page.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

Response for question 1(a)

$(90-60)(.15) + (120-90)(.1) + (135-120)(.05) = 8.25$ gallons

$\int_{60}^{135} f(t)\,dt$ represents the amount of gas pumped, in gallons, from $t=60$ to $t=135$ seconds.

Response for question 1(b)

$f(t)$ is always differentiable, and therefore it must be continuous on $[a, b]$. $a = 60$ & $b = 120$.

According to Rolle's Theorem, if $f(a) = f(b)$ which is true because $f(60) = .1 = f(120)$, there must be some x value, "c", at which $f'(x) = 0$.

Therefore, there must be a value of c such that $f'(c) = 0$

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Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

avg $= \frac{1}{150-0}\int_0^{150} g(t)\,dt = .096$ gallons/second

Response for question 1(d)

$g'(140) = -0.005$ gallons/second/second

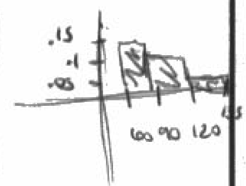
↑ math 8

$g'(140) = -.005$ means that at $t = 140$ seconds, the rate of flow of gasoline is changing at a rate of $-.005$.

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

Answer QUESTION 1 parts (a) and (b) on this page.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0



Response for question 1(a)

$\int_{60}^{135} f(t)\,dt$ in the context of the problem means/establishes the time frame (from 60 to 135 seconds) in which a certain amount of gallons of gasoline are pumped into a gas tank.

$(30 \cdot .15) + (30 \cdot .1) + (30 \cdot .05)$

$4.5 + 3 + 1.5 =$ **9 gallons**

Response for question 1(b)

Yes, there must exist a value of c for $60 < c < 120$ such that $f'(c) = 0$ because in that time, the $f(t)$ fluctuates from .1 gal/s, to .15 gal/sec, and back to .1 gal/s. Given this, at at least 1 instantaneous point in that timeframe, $f'(c)$ must $= 0$ as the rate increases and then decreases.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5216/4

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

$\frac{1}{b-a}\int_a^b f(t)\,dt$

$\frac{1}{150-0}\int_0^{150} \frac{t}{500}\cos\left(\left(\frac{t}{120}\right)^2\right)dt$

$\frac{1}{150}\int_0^{150} \frac{t}{500}\cos\left(\left(\frac{t}{120}\right)^2\right)dt$

$= \boxed{0.0990 \text{ gal/sec}}$

Response for question 1(d)

$g'(140) = ?$ $\quad g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$

$\frac{d}{dt}g(t) = \frac{d}{dt}\left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)_{t=140}$

$= \boxed{-0.00491 \text{ gal/s}^2}$

This figure represents the acceleration at which the gasoline's velocity into the gas tank is functioning - so as gas is pumped, the velocity at which it is pumped decelerates by -0.00491 gal/s².

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0023451

Q5216/5

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

Answer QUESTION 1 parts (a) and (b) on this page.

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

Response for question 1(a)

$\int_{60}^{135} f(t)\,dt$ ~> $15(0.05)+30(0.1)+30(0.15) = 8.25$ g/s

Response for question 1(b)

Yes because c lies in the integral 60<c<120 on the graph

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Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

$g(t) = \left[\frac{t}{500}\right] \cos\left(\left(\frac{t}{120}\right)^2\right)$

$0 \le t \le 150$

$\int_0^{150} \left(\frac{t}{500}\right) \cos\left(\left(\frac{t}{120}\right)^2\right) dt = 14.399$ g/s

Response for question 1(d)

$\int_0^{150} \left(\frac{140}{500}\right) \cos\left(\left(\frac{140}{120}\right)^2\right) dt = 8.742$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0128128

Q5171/5

Question 1

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem students were given a table of times t in seconds and values of a function $f(t)$, which models the rate of flow of gallons of gasoline pumped into a gas tank.

In part (a) students were asked to interpret the meaning of $\int_{60}^{135} f(t)\,dt$ using correct units. Then students were asked to use a right Riemann sum with three subintervals to approximate the value of this integral. A correct response will indicate that the integral represents the accumulated gallons of gasoline pumped into the tank during the time interval from $t = 60$ to time $t = 135$ seconds. The approximation is found using the following expression: $(90 - 60) \cdot f(90) + (120 - 90) \cdot f(120) + (135 - 120) \cdot f(135)$.

In part (b) students were asked to justify whether there must be a value of c, with $60 < c < 120$, such that $f'(c) = 0$. Students are expected to note that because the function f is known to be differentiable on the interval $(0, 150)$, it must be continuous on the subinterval $[60, 120]$. Therefore, because the average rate of change of f on the interval $[60, 120]$ is equal to 0, such a value of c is guaranteed by the Mean Value Theorem.

In part (c) the function $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ was introduced as a second function that modeled the rate of flow of the gasoline. Students were asked to use the model g to find the average rate of flow of the gasoline over the time interval $0 \le t \le 150$. A correct response will show the setup $\frac{1}{150 - 0} \cdot \int_0^{150} g(t)\,dt$ and then use a calculator to find the value 0.096 gallon per second.

In part (d) students were asked to find the value of $g'(140)$ and interpret the meaning of this value in the context of the problem. A correct response will use a calculator to find $g'(140) = -0.005$ and report that at time $t = 140$ seconds the rate at which gasoline is flowing into the tank is decreasing at a rate of 0.005 gallon per second per second.

Sample: 1A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d).

In part (a) the response earned the first point with the statement "the amount of gas pumped, in gallons, from $t = 60$ to $t = 135$ seconds." The response earned the second point for the correct form of the Riemann sum. The response earned the third point for the correct answer.

In part (b) the response earned the first point for "$f(60) = .1 = f(120)$." The response earned the second point because it earned the first point, states that "$f(t)$ is always differentiable, and therefore it must be continuous on $[a, b]$ $a = 60$ & $b = 120$," and states the correct conclusion.

In part (c) the response earned the first point with the inclusion of the average value formula. The response earned the second point with the correct answer.

Question 1 (continued)

In part (d) the response earned the first point for the correct value of $g'(140)$. The response earned the second point with the statement "at $t = 140$ seconds, the rate of flow of gasoline is changing at a rate of $-.005$."

Sample: 1B
Score: 5

The response earned 5 points: 2 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d).

In part (a) the response earned the first point with the statement "the time frame (from 60 to 135 seconds) in which a certain amount of gallons of gasoline are pumped into a gas tank." The response earned the second point for the correct form of the Riemann sum with five of the six factors correct. The third point was not earned because the response contains an error in the Riemann sum.

In part (b) the response did not earn the first point because the expression $f(120) - f(60) = 0$ is not included. Because the first point was not earned, the response is not eligible for the second point. In addition, the response did not earn the second point because the response does not state that f is continuous because f is differentiable.

In part (c) the response earned the first point because the response includes the average value formula. The response earned the second point with the correct answer.

In part (d) the response earned the first point with the presence of the correct value of $g'(140)$. The response did not earn the second point because the response does not interpret the declared value of $g'(140)$ correctly (it needs to discuss a rate of a rate). The words acceleration and velocity should be used to refer to an object in motion.

Sample: 1C
Score: 2

The response earned 2 points: 2 points in part (a), no points in part (b), no points in part (c), and no points in part (d).

In part (a) the response did not earn the first point. The response earned the second point for the correct form of the Riemann sum. The response earned the third point for the correct answer.

In part (b) the response did not earn the first point because the response does not include $f(120) - f(60) = 0$. Because the first point was not earned, the response is not eligible for the second point. In addition, the response did not earn the second point because the response does not state that f is continuous because f is differentiable.

In part (c) the response did not earn the first point because the response does not include the average value formula. The response did not earn the second point because the response does not include the correct answer.

In part (d) the response did not earn the first point because the response does not include the value of $g'(140)$. The response did not earn the second point because the response does not include the correct interpretation.

2023

AP

AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 2

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

Part A (AB): Graphing calculator required
Question 2 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Stephen swims back and forth along a straight path in a 50-meter-long pool for 90 seconds. Stephen's velocity is modeled by $v(t) = 2.38e^{-0.02t}\sin\left(\frac{\pi}{56}t\right)$, where t is measured in seconds and $v(t)$ is measured in meters per second.

	Model Solution	Scoring	
(a)	Find all times t in the interval $0 < t < 90$ at which Stephen changes direction. Give a reason for your answer.		
	For $0 < t < 90$, $v(t) = 0 \Rightarrow t = 56$.	Considers sign of $v(t)$	**1 point**
	Stephen changes direction when his velocity changes sign. This occurs at $t = 56$ seconds.	Answer with reason	**1 point**

Scoring notes:

- A response that considers $v(t) = 0$ earns the first point.
- A response of "Stephen changes direction when his velocity changes sign" earns the first point for considering the sign of $v(t)$ but must include the answer of $t = 56$ in order to earn the second point.
- A response of $t = 56$ with no supporting work does not earn either point.
- Any presented values of t outside the interval $0 < t < 90$ will not affect scoring.

Total for part (a) — 2 points

	Model Solution	Scoring	
(b)	Find Stephen's acceleration at time $t = 60$ seconds. Show the setup for your calculations, and indicate units of measure. Is Stephen speeding up or slowing down at time $t = 60$ seconds? Give a reason for your answer.		
	$v'(60) = a(60) = -0.0360162$	$a(60)$ with setup	**1 point**
	Stephen's acceleration at time $t = 60$ seconds is -0.036 meter per second per second.	Acceleration units	**1 point**

$v(60) = -0.1595124 < 0$ Stephen is speeding up at time $t = 60$ seconds because Stephen's velocity and acceleration are both negative at that time.	Speeding up with reason	**1 point**

Scoring notes:

- The minimum work needed to earn the first point is $v'(60) = -0.036$.
  - $a(60) = -0.0360162$ is not sufficient to earn the first point. The connection $v'(t) = a(t)$ or $v'(60) = a(60)$ must be explicitly shown.
- A response must declare a value for $a(60)$ to be eligible for the second point.
- In order to earn the third point the presented conclusion must be consistent with a negative velocity at time $t = 60$ and the presented value of $a(60)$.
- A response does not need to find the value of $v(60)$; an implied sign is sufficient.
  - The statement "Stephen is speeding up because $a(60)$ and $v(60)$ have the same sign" (or equivalent) earns the third point, provided a negative value is presented for $a(60)$.
- A response that reports an incorrect sign or value of $v(60)$ does not earn the third point. Any presented value of $v(60)$ must be correct for the number of digits presented, from one up to three decimal places in order to earn the third point.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode).
  - In degree mode, there are two possible values for $v'(60)$. A response that declares $v'(60) = -0.000141$ does not earn the first point but would earn the third point in the presence of $v(60) = 0.042089$ or $v(60) > 0$ and the conclusion that Stephen is slowing down.
  - Similarly, a response that declares $v'(60) = 0.039304$ does not earn the first point but would earn the third point in the presence of $v(60) = 0.042089$ or $v(60) > 0$ and the conclusion that Stephen is speeding up.

Total for part (b) 3 points

(c) Find the distance between Stephen's position at time $t = 20$ seconds and his position at time $t = 80$ seconds. Show the setup for your calculations.

$\int_{20}^{80} v(t)\, dt$	Integral	**1 point**
$= 23.383997$ The distance between Stephen's positions at $t = 20$ seconds and $t = 80$ seconds is 23.384 (or 23.383) meters.	Answer	**1 point**

Scoring notes:

- The first point is earned only for $\int_{20}^{80} v(t)\,dt$ (or the equivalent) with or without the differential.
- The second point is earned only for an answer of 23.384 (or 23.383) regardless of whether the first point was earned.
- Degree mode: In degree mode, $\int_{20}^{80} v(t)\,dt = 2.407982$.

Total for part (c) 2 points

(d) Find the total distance Stephen swims over the time interval $0 \le t \le 90$ seconds. Show the setup for your calculations.

$\int_0^{90} \lvert v(t)\rvert\,dt$	Integral	**1 point**
$= 62.164216$ The total distance Stephen swims over the time interval $0 \le t \le 90$ seconds is 62.164 meters.	Answer	**1 point**

Scoring notes:

- The first point is earned only for $\int_0^{90} \lvert v(t)\rvert\,dt$ or $\int_0^{56} v(t)\,dt - \int_{56}^{90} v(t)\,dt$ (or the equivalent), with or without the differential(s).
- The second point is earned only for an answer of 62.164 regardless of whether the first point was earned.
- Degree mode: In degree mode, $\int_0^{90} \lvert v(t)\rvert\,dt = 3.127892$.

Total for part (d) 2 points

Total for question 2 9 points

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

Changes directions
$v(t) = 0$ and changes signs

$t = 56$ second

At $t = 56$ seconds Stephen changes directions because $v(t) = 0$ and changes signs.

Response for question 2(b)

$v'(t) = a(t)$

$v'(t)\Big|_{x=60} = -0.0360$ meters/second^2

$a(60) = -0.0360$ meters/second^2

$v(60) = -0.1595$ meters/second

Stephen is speeding up at $t = 60$ seconds because $a(t)$ and $v(t)$ have the same sign.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Sample 2A

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$$\int_{20}^{80} v(t)\, dt = 23.384 \text{ meters}$$

The distance between stephen's position at $t = 20$ sec and position at $t = 80$ sec. is 23.384 meters.

Response for question 2(d)

$$\int_{0}^{90} |v(t)|\, dt = 62.164 \text{ meters}$$

Stephen swims a total distance of 62.164 meters over the time interval $0 \le t \le 90$ seconds.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0031634

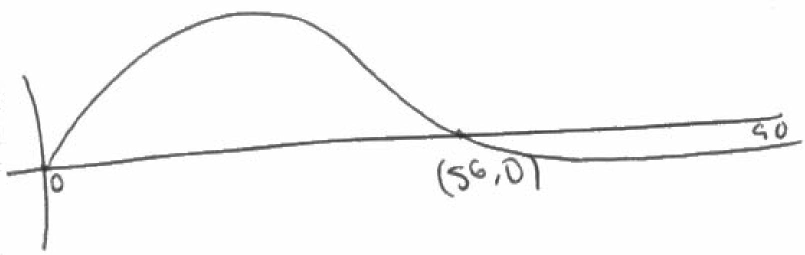
Q5218/7

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

The only time that stephen changes direction is when t= 56 wich means that Stephen changes his velocity from being positive to being negative



Response for question 2(b)

$v(t) = 2.38e^{-0.02t} \sin\left(\frac{\pi}{56}x\right)$

$v(60) = -0.159512$ m/s

$v'(60) = -0.636016$ m/s²

Stephen at the time of 60 second he is speeding up towards the negative direction because v(60) is negative and v'(60) is also negative

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$\int_0^{20} v(t)dt = 18.5887$ $\int_0^{80} v(t) = 41.4437$

$\int_0^{20} v(t)dt = 18.5887$m meters the position at time 20 is equal to 18.5887

$\int_0^{80} v(t)dt = 41.4437$ the position at time 80 is 41.4437 meters

the distance between those position is 23.0892 meters

Response for question 2(d)

The total distance $\int_0^{90} f(x)\,dx$ is 37.2041 m

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0140107

Q5171/7

Sample 2C

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

$v(t) = 2.38e^{-0.02t}\sin\left(\frac{\pi}{56}t\right)$

~~t~~ = 52.273

t = 52.273

v(t) changes from positive to negative

Response for question 2(b)

$v'(t) = -0.467$

Stephen is ~~speeding up~~ slowing down because his velocity is positive while his acceleration is negative.

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$v(t) = 2.38e^{-0.02t}\sin\left(\frac{\pi}{56}t\right)$

$\int_0^{80}\left(2.38e^{-0.02t}\sin\left(\frac{\pi}{56}t\right)\right)dt - \int_0^{20}\left(2.38e^{-0.02t}\sin\left(\frac{\pi}{56}t\right)\right)dt$

Response for question 2(d)

$\int_0^{90} 2.38e^{-0.02t}\sin\left(\frac{\pi}{56}(t)\right)$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0118433

Q5216/7

Question 2

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem students were told that Stephen is swimming back and forth along a straight path in a 50-meter pool for 90 seconds with a velocity modeled by the function $v(t) = 2.38e^{-0.02t}\sin\left(\frac{\pi}{56}t\right)$. Here t is measured in seconds and $v(t)$ is measured in meters per second.

In part (a) students were asked to find all times t in the interval $0 < t < 90$ at which Stephen changes direction. A correct response will indicate understanding that Stephen changes direction when his velocity changes sign. Based on the model for velocity, it is clear that $v(56) = 0,$ and a calculator could be used to confirm that $v(t)$ does change sign at this time.

In part (b) students were asked to find Stephen's acceleration at time $t = 60$ seconds and to indicate units of measure. Then students were asked to provide a reason for whether Stephen was speeding up or slowing down at this time. A correct response will communicate that Stephen's acceleration is the derivative of his velocity and will use a calculator to find $a(60) = v'(60) = -0.036$ meter per second per second. Then the response will evaluate the given velocity function at $t = 60$ in order to determine that $v(60)$ is also negative and conclude that, because the velocity and acceleration have the same sign at $t = 60,$ Stephen must be speeding up.

In part (c) students were asked to find the distance between Stephen's position at time $t = 20$ seconds and his position at time $t = 80$ seconds. A correct response will show the calculator setup $\int_{20}^{80} v(t)\,dt,$ with a numerical answer of 23.384 meters.

In part (d) students were asked to find the total distance Stephen swims over the time interval from $t = 0$ to $t = 90$ seconds. A correct response will provide the setup $\int_{0}^{90} |v(t)|\,dt$ and use a calculator to find the total distance is 62.164 meters.

Sample: 2A
Score: 9

The response earned 9 points: 2 points in part (a), 3 points in part (b), 2 points in part (c), and 2 points in part (d).

In part (a) the response earned the first point with the equation $v(t) = 0$ in line 2. The statements, "changes directions," "$v(t) = 0$ and changes signs," and "$t = 56,$" in lines 1, 2, 3, and 4 would have earned the second point; a summary statement is not required. In this case, the response correctly states, "At $t = 56$ seconds Stephen changes directions because $v(t) = 0$ and changes signs," so the second point was earned.

In part (b) the response earned the first point with the answer of -0.0360 in line 2, given that the response explicitly makes the connection $v'(t) = a(t)$ in line 1. The use of $x = 60$ instead of $t = 60$ in line 2 does not affect scoring. The second point was earned with the acceleration units of $\text{meters}/\text{second}^2$ in line 2. The response would have earned the third point with the sentence, "Stephen is speeding up at $t = 60$ seconds because $a(t)$ and $v(t)$ have the

Question 2 (continued)

same sign," without declaring a numerical value of $v(60)$. In this case, the response correctly declares $v(60) = -0.1595$ in line 4, so the third point was earned.

In part (c) the response earned the first point with the definite integral $\int_{20}^{80} v(t)\, dt$ in line 1. The answer of 23.384 in line 1 would have earned the second point; a summary statement is not required. In this case, the response correctly states, "The distance between Stephen's position at $t = 20$ sec and position at $t = 80$ sec. is 23.384 meters," so the second point was earned.

In part (d) the response earned the first point with the definite integral $\int_{0}^{90} |v(t)|\, dt$ in line 1. The response would have earned the second point with the answer of 62.164 in line 1; a summary statement is not required. In this case, the response correctly states, "Stephen swims a total distance of 62.164 meters over the time interval $0 \le t \le 90$ seconds," so the second point was earned.

Sample: 2B
Score: 5

The response earned 5 points: 2 points in part (a), 3 points in part (b), no points in part (c), and no points in part (d).

In part (a) the response earned the first point for the statement, "Stephen changes his velocity from being positive to being negative," in lines 2, 3, and 4. The second point was earned for the answer of $t = 56$ in the presence of correct reasoning.

In part (b) the response earned the first point with $v'(60) = -0.036016$ in line 3. The second point was earned for the correct acceleration units of $^{\text{m}}/_{\text{s}^2}$ in line 3. The response would have earned the third point with the statement, "he is speeding up towards the negative direction because $v(60)$ is negative and $v'(60)$ is also negative." A declared value for $v(60)$ is not required; an implied sign is sufficient. In this case, the response correctly declares that $v(60) = -0.159512$ in line 2, so the third point was earned.

In part (c) the response did not earn the first point because the definite integral $\int_{20}^{80} v(t)\, dt$ (or the equivalent) is not presented, and the declared answer of 23.0842 in line 4 is not the difference of the two presented values $\int_{0}^{80} v(t)\, dt = 41.4437$ and $\int_{0}^{20} v(t)\, dt = 18.5887$ in line 1. If the answer presented were the difference of the presented values of $\int_{0}^{80} v(t)\, dt$ and $\int_{0}^{20} v(t)\, dt,$ the response would have earned the first point. The second point was not earned because the answer 23.0892 in line 4 is incorrect.

In part (d) the response did not earn the first point because the presented definite integral is incorrect. The second point was not earned because the answer 37.2041 in line 1 is incorrect.

Sample: 2C
Score: 2

The response earned 2 points: 1 point in part (a), no points in part (b), 1 point in part (c), and no points in part (d).

Question 2 (continued)

In part (a) the response earned the first point with the statement, "$v(t)$ changes from positive to negative," in line 4. The second point was not earned because the boxed answer of 52.273 is incorrect.

In part (b) the response did not earn the first point because the declared answer of -0.467 is incorrect. Even in the presence of the correct answer, the presentation of $v'(t)$ instead of $v'(60)$ would not have earned the first point. However, a response that reports a value for $v'(t)$ instead of $v'(60)$ is still eligible for both the second and third points. The second point was not earned because units for acceleration are not declared. The response incorrectly states that velocity is positive in line 3, so the third point was not earned.

In part (c) the response earned the first point with the expression:
$\int_0^{80} \left(2.38e^{-0.02t} \sin\left(\frac{\pi}{56}t\right)\right) dt - \int_0^{20} \left(2.38e^{-0.02t} \sin\left(\frac{\pi}{56}t\right)\right) dt.$ The response does not present an answer, so the second point was not earned.

In part (d) the response did not earn the first point because the integral presented is incorrect due to the absence of absolute value bars in the integrand. The missing differential in the integrand does not affect scoring. The response does not present an answer, so the second point was not earned.

2023

AP

AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 3

☑ Scoring Guidelines

☑ Student Samples

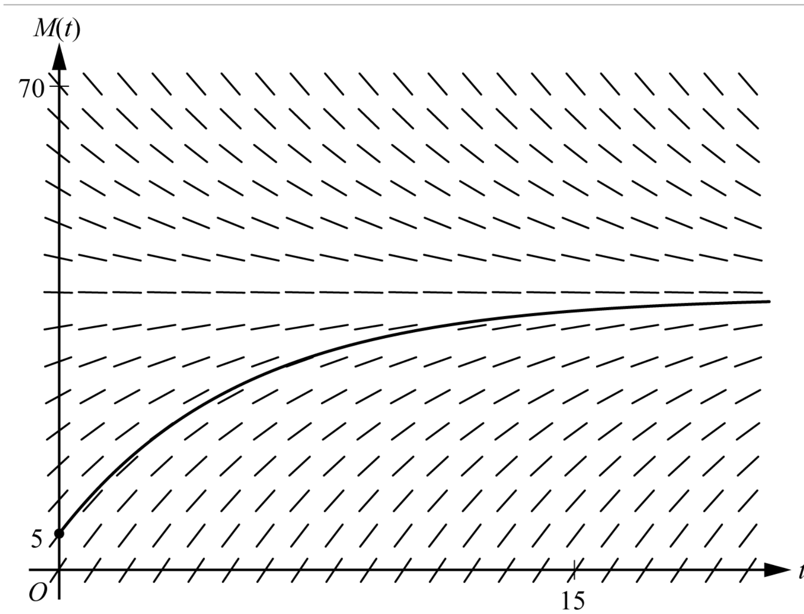
☑ Scoring Commentary

Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($°C$) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is $5°C$. It can be shown that $M(t) < 40$ for all values of t.

	Model Solution	Scoring	
(a)	A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.		
		Solution curve	**1 point**

Scoring notes:

- The solution curve must pass through the point $(0, 5)$, extend reasonably close to the left and right edges of the rectangle, and have no obvious conflicts with the given slope lines.
- Only portions of the solution curve within the given slope field are considered.
- The solution curve must lie entirely below the horizontal line segments at $M = 40$.

Total for part (a) — 1 point

(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

$\left.\frac{dM}{dt}\right\rvert_{t=0} = \frac{1}{4}(40 - 5) = \frac{35}{4}$	$\left.\frac{dM}{dt}\right\rvert_{t=0}$	**1 point**
The tangent line equation is $y = 5 + \frac{35}{4}(t - 0)$. $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$ The temperature of the milk at time $t = 2$ minutes is approximately 22.5° Celsius.	Approximation	**1 point**

Scoring notes:

- The value of the slope may appear in a tangent line equation or approximation.
- A response of $5 + \frac{35}{4} \cdot 2$ is the minimal response to earn both points.
- A response of $\frac{1}{4}(40 - 5)$ earns the first point. If there are any subsequent errors in simplification, the response does not earn the second point.
- In order to earn the second point the response must present an approximation found by using a tangent line that:
  - passes through the point $(0, 5)$ and
  - has slope $\frac{35}{4}$ or a nonzero slope that is declared to be the value of $\frac{dM}{dt}$.
- An unsupported approximation does not earn the second point.
- The approximation need not be simplified, but the response does not earn the second point if the approximation is simplified incorrectly.

Total for part (b) 2 points

(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

$\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right) = -\frac{1}{16}(40 - M)$	$\frac{d^2M}{dt^2}$	**1 point**
Because $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down. Therefore, the tangent line approximation of $M(2)$ is an overestimate.	Overestimate with reason	**1 point**

Scoring notes:

- The first point is earned for either $\frac{d^2M}{dt^2} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right)$ or $\frac{d^2M}{dt^2} = -\frac{1}{16}(40 - M)$ (or equivalent). A response that presents any subsequent simplification error does not earn the second point.
- A response that presents an expression for $\frac{d^2M}{dt^2}$ in terms of $\frac{dM}{dt}$ but fails to continue to an expression in terms of M (i.e., $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt}$) does not earn the first point but is eligible for the second point.
- If the response presents an expression for $\frac{d^2M}{dt^2}$ that is incorrect, the response is eligible for the second point only if the expression is a nonconstant linear function that is negative for $5 < M < 40$.
  - Special case: A response that presents $\frac{d^2M}{dt^2} = \frac{1}{16}(40 - M)$ does not earn the first point but is eligible to earn the second point for a consistent answer and reason.
- To earn the second point a response must include $\frac{d^2M}{dt^2} < 0$, or $\frac{dM}{dt}$ is decreasing, or the graph of M is concave down, as well as the conclusion that the approximation is an overestimate.
- A response that presents an argument based on $\frac{d^2M}{dt^2}$ or concavity at a single point does not earn the second point.

Total for part (c) 2 points

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

$\frac{dM}{dt} \cdot$ $\frac{dM}{40 - M} = \frac{1}{4}dt$ $\int \frac{dM}{40 - M} = \int \frac{1}{4}\,dt$	Separates variables	**1 point**
$-\ln\lvert 40 - M \rvert = \frac{1}{4}t + C$	Finds antiderivatives	**1 point**
$-\ln\lvert 40 - 5 \rvert = 0 + C \Rightarrow C = -\ln 35$ $M(t) < 40 \Rightarrow 40 - M > 0 \Rightarrow \lvert 40 - M \rvert = 40 - M$ $-\ln(40 - M) = \frac{1}{4}t - \ln 35$ $\ln(40 - M) = -\frac{1}{4}t + \ln 35$	Constant of integration and uses initial condition	**1 point**

$40 - M = 35e^{-t/4}$ $M = 40 - 35e^{-t/4}$	Solves for M	**1 point**

Scoring notes:

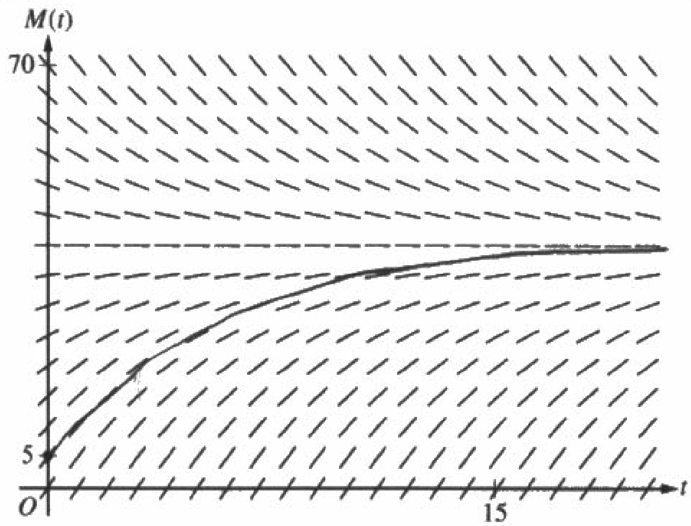
- A response with no separation of variables earns 0 out of 4 points.
- A response that presents an antiderivative of $-\ln(40 - M)$ without absolute value symbols is eligible for all 4 points.
- A response with no constant of integration can earn at most the first 2 points.
- A response is eligible for the third point only if it has earned the first 2 points.
    - Special Case: A response that presents $+\ln(40 - M) = \frac{t}{4} + C$ (or equivalent) does not earn the second point, is eligible for the third point, but not eligible for the fourth.
- An eligible response earns the third point by correctly including the constant of integration in an equation and substituting 0 for t and 5 for M.
- A response is eligible for the fourth point only if it has earned the first 3 points.
- A response earns the fourth point only for an answer of $M = 40 - 35e^{-t/4}$ or equivalent.

Total for part (d) 4 points

Total for question 3 9 points

Answer QUESTION 3 parts (a) and (b) on this page.

Response for question 3(a)



Response for question 3(b)

$$\left.\frac{dM}{dt}\right|_{t=0} = \frac{1}{4}(40-5) = \frac{35}{4}$$

$$(M-5) = \frac{35}{4}(t-0)$$

$$M = \frac{35}{4}t + 5$$

$$M(2) \approx \frac{35}{4}\cdot 2 + 5$$

$$= \frac{35}{2} + 5$$

$$= \boxed{\frac{45}{2}\,^\circ C}$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Sample 3A

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

$$\frac{d^2M}{dt^2} = \frac{d}{dt}\left[\frac{1}{4}(40-M)\right]$$

$$= \frac{1}{4}\left(0 - \frac{dM}{dt}\right)$$

$$= \boxed{-\frac{1}{16}(40-M)}$$

Since $\frac{d^2M}{dt^2}$ value is always negative for all t ($\because M(t) < 40$ for all t), the graph is always concave down. Thus, the approximation is overestimate.

Response for question 3(d)

$$\int \frac{1}{40-M}\, dM = \int \frac{1}{4}\, dt$$

$$-\ln|40-M| = \frac{1}{4}t + C$$

$$40 - M = Ce^{-\frac{1}{4}t}$$

$$M = Ce^{-\frac{1}{4}t} + 40$$

$$5 = C \cdot e^0 + 40$$

$$\rightarrow C = -35$$

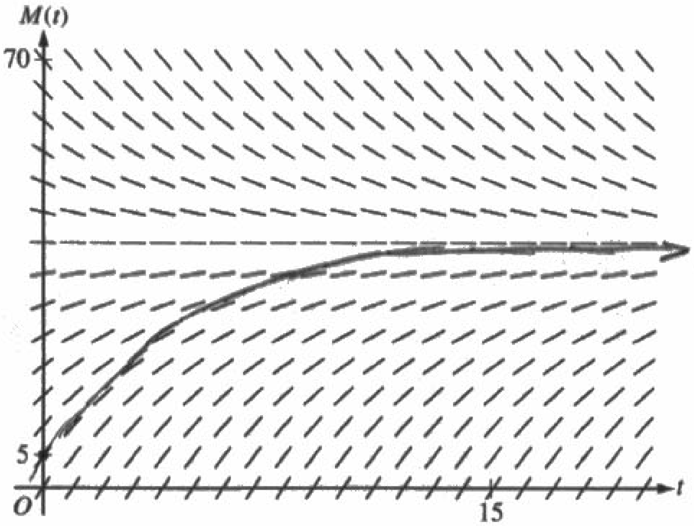
$$\therefore M = -35e^{-\frac{1}{4}t} + 40$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5178/9

Answer QUESTION 3 parts (a) and (b) on this page.

Response for question 3(a)



Response for question 3(b)

$\frac{dM}{dt} = \frac{1}{4}(40-M) \Rightarrow \frac{1}{4}(40-5) = \frac{35}{4}$ ↙ slope

$M = 5$ at $t = 0$

$5 + \frac{70}{4} = \boxed{\frac{90}{4}\ C^\circ}$

$5 + \left(\frac{35}{4} \cdot 2\right) =$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5171/8

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

c) $\frac{d^2M}{dt^2} = \frac{-M}{4} \rightarrow (-)$ + therefore $M(x)$ is always concave down

$M(2)$ from part (b) is an over estimation bc actual $M(2)$ is less than $\frac{90}{4}$ C° ⇒ $M(x)$ is concave down and $M'(x)$ is getting closer + closer to 0. Therefore, a tangent line won't account for the ever decreasing nature of $M'(x)$ + will overestimate

Response for question 3(d)

$u = 40 - M$
$-du = dM$

$$\int \frac{4}{40-M}\, dM = \int dt$$

$-\ln|40-M| + c = t$ ⟶ $e^{-\ln|40-5| + c} = e^{0}$

$-35 + e^{c} = 1$

$\ln e^{c} = \ln 36$

$c = \ln 36$

$t = -\ln|40-M| + \ln 36$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

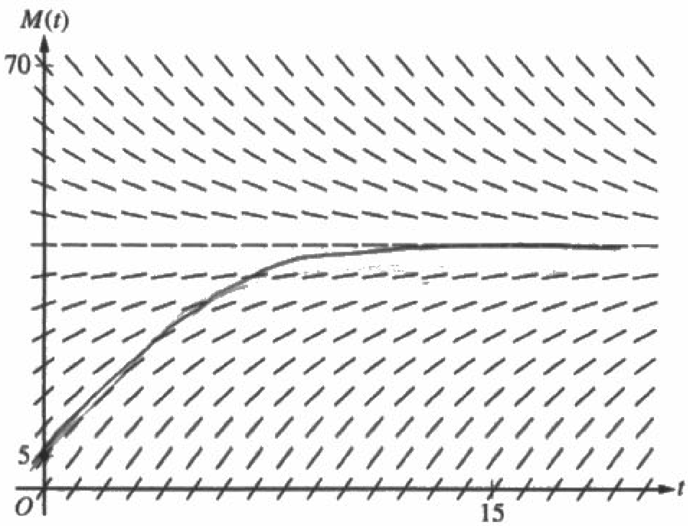
3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (a) and (b) on this page.

Response for question 3(a)

$\frac{1}{4}(40-5)$

10



Response for question 3(b)

t = mins since bottle placed

M(t) = milk temp at time t

$\frac{dM}{dt} = \frac{1}{4}(40-M)$

initial t = 5°C

$f'(0) = \frac{1}{4}(40-M)$

$= \frac{1}{4}(40-5)$

$= \frac{35}{4}$

$f(a) + f'(a)(x-a)$

$f(0) + f'(0)(2-0) \approx M(2)$

$5 + \frac{35}{4}(2)$

$5 + \frac{35}{2}$

$\frac{10}{2} + \frac{35}{2}$

$\frac{45}{2}$

23.5 °C ≈ temperature of milk at t = 2

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5178/8

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

$\frac{40-M}{4} \rightarrow \frac{4\left(-\frac{dM}{dt}\right) - (40-M)(0)}{16}$

$\frac{4\left(-\frac{dM}{dt}\right)}{16} = \frac{4\left(\frac{40-M}{4}\right)}{16} = \frac{40-M}{16} = \frac{d^2M}{dt^2}$

$\frac{40-M}{16}$ at $t=2$

is positive at $t=2$

thus, the graph is concave down (2nd derivative / the) meaning that it is an overestimate for the actual value of M(2)

Response for question 3(d)

$M(t)$

$\frac{1}{4}(40-M) = \frac{dM}{dt}$

$40-M = \frac{4dM}{dt}$

$dt(40-M) = 4dM$

$dt = \frac{4dM}{40-M}$

$dt = dM \cdot \frac{1}{10-M}$

$t = -\ln(10-M) + C$

$0 = -\ln(10-5) + C$

$0 = \ln 5 + C$

$-\ln 5 = C$

$t = -\ln(10-M) - \ln 5$

Question 3

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem students were told that an increasing function M models the temperature of a bottle of milk taken out of the refrigerator and placed in a pan of hot water to warm. The function M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$, where t is measured in minutes since the bottle was placed in the pan and M is measured in degrees Celsius. At time $t = 0$, the temperature of the milk is $5°C$.

In part (a) students were given a slope field for the differential equation and asked to sketch the solution curve through the point $(0, 5)$. A correct response will draw a curve that passes through the point $(0, 5)$, follows the indicated slope segments, extends reasonably close to the left and right edges of the slope field, and lies entirely below the horizontal line segments at $M = 40$.

In part (b) students were asked to use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$. A correct response will find the slope of the tangent line when $t = 0$ is $\left.\frac{dM}{dt}\right|_{t=0} = \frac{35}{4}$ and then use the tangent line equation, $y = 5 + \frac{35}{4}t$, to find that $M(2) \approx 22.5$.

In part (c) students were asked to find an expression for $\frac{d^2M}{dt^2}$ in terms of M and then to use $\frac{d^2M}{dt^2}$ to reason whether the approximation from part (b) is an underestimate or overestimate for the actual value of $M(2)$. A correct response will differentiate the given differential equation to obtain $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{16}(40 - M)$, then use the information that $M(t) < 40$ to determine that the second derivative of M is negative and therefore the graph of M is concave up and the approximation in part (b) is an overestimate.

In part (d) students were asked to use separation of variables to find an expression for the particular solution to the given differential equation with initial condition $M(0) = 5$. A correct response will separate the variables, integrate, use the initial condition to find the constant of integration, and arrive at a solution of $M = 40 - 35e^{-t/4}$.

Sample: 3A
Score: 9

The response earned 9 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 4 points in part (d).

In part (a) the response earned the point for the solution curve.

In part (b) the response earned the first point by stating $\frac{1}{4}(40 - 5)$ on the first line. The response correctly simplifies this expression, but this simplification is not needed. The second point was earned for the expression $\frac{35}{4} \cdot 2 + 5$ on the fourth line. The response simplifies the expression correctly as $\frac{45}{2}$, however, this is not needed to earn the second point.

Question 3 (continued)

In part (c) the response earned the first point for the boxed expression $-\frac{1}{16}(40 - M)$ on the third line on the left. The second point was earned for the explanation given on the fourth, fifth, and sixth lines.

In part (d) the response earned the first point for the correct separation on the first line. The second point was earned for the correct antiderivatives on the second line. The third point was earned on the fifth line on the left for the equation $5 = C \cdot e^0 + 40.$ The fourth point was earned for solving for M on the seventh line.

Sample: 3B
Score: 6

The response earned 6 points: 1 point in part (a), 2 points in part (b), 1 point in part (c), and 2 points in part (d).

In part (a) the response earned the point for the solution curve.

In part (b) the response earned the first point by stating $\frac{1}{4}(40 - 5)$ on the first line. The response then correctly simplifies this expression to $\frac{35}{4},$ but this simplification is not needed. The second point was earned for the expression $5 + \frac{70}{4}$ on the third line. The boxed expression, though correct, is not needed to earn the second point.

In part (c) the response did not earn the first point because the expression for $\frac{d^2M}{dt^2}$ is not correct. The response is eligible for the second point because the expression for $\frac{d^2M}{dt^2}$ is a nonconstant linear function that is negative for $5 < M < 40.$ The second point was earned for the statement on the first through third lines. On the third through sixth lines, the response makes statements that are correct but not needed to earn the second point.

In part (d) the response earned the first point for the correct separation on the third line on the left. The second point was not earned because the antiderivative of $\frac{4}{40 - M}$ is not correct. However, the response is eligible to earn the third point. The third point was earned on the first line on the right for the equation $-\ln|40 - 5| + C = 0.$ The response is not eligible for the fourth point because the response did not earn all of the first three points in this part.

Sample: 3C
Score: 2

The response earned 2 points: no points in part (a), 1 point in part (b), no points in part (c), and 1 point in part (d).

In part (a) the response did not earn the point because the solution curve does not lie entirely below the horizontal segments.

In part (b) the response earned the first point for the expression $\frac{1}{4}(40 - 5)$ on the sixth line on the left. The response did not earn the second point because the final approximation is not simplified correctly on the seventh line on the right.

Question 3 (continued)

In part (c) the response did not earn the first point because the expression for $\frac{d^2M}{dt^2}$ is not correct. The response provides a form of the only positive second derivative that is eligible for the second point, but it did not earn the second point because both an incorrect conclusion and local argument are made.

In part (d) the response earned the first point for the correct separation in the fifth line on the left. The second point was not earned because the antiderivative is not correct due to a simplification error prior to integration. The presented antiderivative is not eligible to earn the third or fourth points.

2023

AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 4

☑ Scoring Guidelines

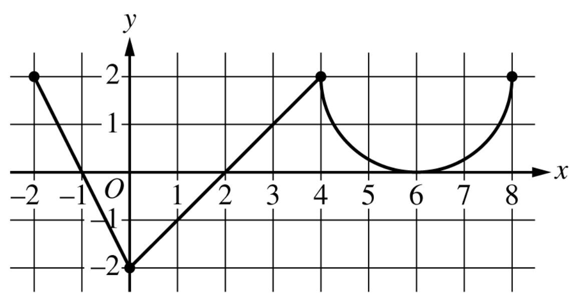
☑ Student Samples

☑ Scoring Commentary

Part B (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



Graph of f'

The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.

	Model Solution	Scoring	
(a)	Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.		
	$f'(x) > 0$ on $(2, 6)$ and $f'(x) > 0$ on $(6, 8)$. $f'(x)$ does not change sign at $x = 6$, so there is neither a relative maximum nor a relative minimum at this location.	Answer with reason	**1 point**

Scoring notes:

- A response that declares $f'(x)$ does not change sign at $x = 6$, so neither, is sufficient to earn the point.
- A response does not have to present intervals on which $f'(x)$ is positive or negative, but if any are given, they must be correct. Any presented intervals may include none, one, or both endpoints.
- A response that declares $f'(x) > 0$ before and after $x = 6$ does not earn the point.

Total for part (a) 1 point

(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

The graph of f is concave down on $(-2, 0)$ and $(4, 6)$ because f' is decreasing on these intervals.	Intervals	**1 point**
	Reason	**1 point**

Scoring notes:

- The first point is earned only by an answer of $(-2, 0)$ and $(4, 6)$, or these intervals including one or both endpoints.
- A response must earn the first point to be eligible for second point.
- To earn the second point a response must correctly discuss the behavior of f' or the slopes of f'.
- Special case: A response that presents exactly one of the two correct intervals with a correct reason earns 1 out of 2 points.

Total for part (b) 2 points

(c) Find the value of $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

Because f is differentiable at $x = 2$, f is continuous at $x = 2$, so $\lim_{x \to 2} f(x) = f(2) = 1$. $\lim_{x \to 2} (6f(x) - 3x) = 6 \cdot 1 - 3 \cdot 2 = 0$ $\lim_{x \to 2} \left(x^2 - 5x + 6\right) = 0$	Limits of numerator and denominator	**1 point**
Because $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$ is of indeterminate form $\frac{0}{0}$, L'Hospital's Rule can be applied. Using L'Hospital's Rule, $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x \to 2} \frac{6f'(x) - 3}{2x - 5} = \frac{6 \cdot 0 - 3}{2 \cdot 2 - 5} = 3.$	Uses L'Hospital's Rule	**1 point**
	Answer	**1 point**

Scoring notes:

- The first point is earned by the presentation of two separate limits for the numerator and denominator.
- A response that presents a limit explicitly equal to $\frac{0}{0}$ does not earn the first point.
- The second point is earned by applying L'Hospital's Rule, that is, by presenting at least one correct derivative in the limit of a ratio of derivatives.
- The third point is earned for the correct answer with supporting work.

Total for part (c) 3 points

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

$f'(x) = 0 \Rightarrow x = -1,\ x = 2,\ x = 6$	Considers $f'(x) = 0$	**1 point**
The function f is continuous on $[-2, 8]$, so the candidates for the location of an absolute minimum for f are $x = -2,\ x = -1,\ x = 2,\ x = 6,$ and $x = 8.$	Justification	**1 point**
	Answer	**1 point**

x	$f(x)$
-2	3
-1	4
2	1
6	$7 - \pi$
8	$11 - 2\pi$

The absolute minimum value of f is $f(2) = 1$.

Scoring notes:

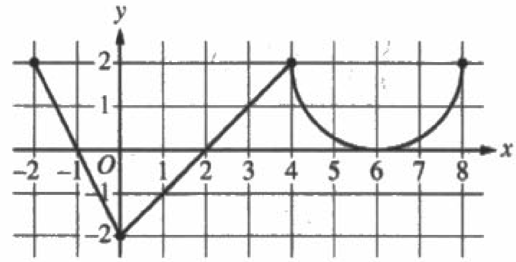
- To earn the first point a response must state $f' = 0$ or equivalent. Listing the zeros of f' is not sufficient.
- A response that presents any error in evaluating f at any critical point or endpoint will not earn the justification point.
- A response need not present the value of $f(-1)$ provided $x = -1$ is eliminated because it is the location of a local maximum. A response need not present the value of $f(6)$ provided $x = 6$ is eliminated by reference to part (a) or eliminated through analysis.
- A response need not present the value of $f(8)$ provided there is a presentation that argues $f'(x) \ge 0$ for $x > 2$ and, therefore, $f(8) > f(2)$.
- A response that does not consider both endpoints does not earn the justification point.
- The answer point is earned only for indicating that the minimum value is 1. It is not earned for noting that the minimum occurs at $x = 2$.

Total for part (d) 3 points

Total for question 4 9 points

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f'

Response for question 4(a)

f has neither a relative max. or min. at x=6 since there is no sign change on f' at x=6.

Response for question 4(b)

f is concave down on (-2,0) and (4,6) because f'(x) is decreasing on those intervals

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0027329

Q5216/10

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6}$

$\lim_{x\to 2} 6f(x)-3x = 0$

$\lim_{x\to 2} x^2-5x+6 = 0$

Since f is differatiable and therefore contious at x = 2
L'Hospital's Rule applys

$\lim_{x\to 2} \frac{6f'(x)-3}{2x-5} = \frac{-3}{-1} = \boxed{3}$

Response for question 4(d)

1 −

$\frac{1}{4}\pi 4$ π

$4-\pi$

$\frac{1}{2}\pi r^2$ 2π

$8-2\pi$

Critical values
$f'(x)=0$ at.
−1, 2, 6

x	f(x)
−2	$1-\int_{-2}^{2} f(x)\,dx = 1-(-2) = 3$
−1	$1-\int_{-1}^{2} f(x)\,dx = 1--3 = 4$
2	1
6	$1+\int_{2}^{6} f(x)\,dx = 1+2+(4-\pi) = 7-\pi$
8	$1+\int_{2}^{8} f(x)\,dx = 1+2+(8-2\pi) = 11-2\pi$

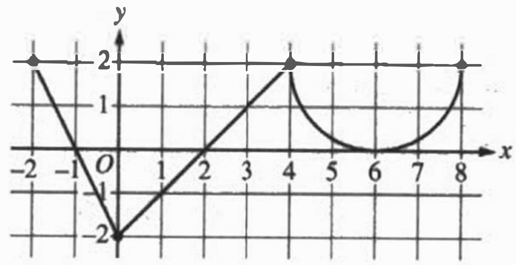
The absolute min value of f on [−2, 8] is at x = 2 and is equal to 1.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5216/11

4 4 4 4 4 **NO CALCULATOR ALLOWED** 4 4 4 4 4

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f'

Response for question 4(a)

At $x=6$, f does not have relative minimum nor relative maximum because f' does not change sign, meaning f does not change from ~~that~~ increasing to decreasing nor decreasing to increasing.

Response for question 4(b)

f is concave down on the interval $(-2,0) \cup (4,6)$ because f' is decreasing, meaning f'' is negative.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0015407

Q5171/10

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$\lim_{x\to 2}\frac{6f(x)-3x}{x^2-5x+6} = \frac{6f(2)-3\times 2}{2^2-5\times 2+6} = \frac{6\times 1-6}{4-10+6} = \frac{0}{0}$

$\lim_{x\to 2}\frac{6f'(x)-3}{2x-5} = \frac{6f'(2)-3}{2\times 2-5} = \frac{6\times 0-3}{4-5} = \frac{-3}{-1} = 3$

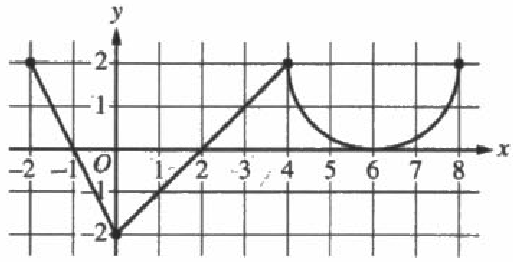
Response for question 4(d)

The absolute minimum value is at $x=2$ because f' changes from negative to positive. ←

$f(2)=1$ The absolute minimum value is 1.

Also, $f(-2)=f(0)$ because the positive area and negative area canceled out, and f kept dereasing on $(0,2)$ according to the negative f' on the interval, so absolute.

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f'

$f(2) = 1$

Response for question 4(a)

Neither. Since $f'(6)=0$, f does have an extrema at $x=6$, but since $f''(6)=0$, as well, $f(6)$ must be an inflection point.

Response for question 4(b) For $f(x)$ to be concave down, $f''(x)$ must be less than 0. Based on the given graph of f', the graph of f is concave down on the intervals $-2<x<0$ and $4<x<6$ because on these intervals, $f''(x)<0$.

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$f(x) = x-2,\ 0 \le x \le 4$

$\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6} = \lim_{x\to 2} \frac{3(2f(x)-x)}{(x-3)(x-2)} \rightarrow \frac{2f'(2)}{(x-3)} = 0$

Response for question 4(d)

Min → $f'(x)=0,\ f''(x)>0$

$f'(2)=0$

$f''(2)=1 \rightarrow 1>0$

$f(2)=1$

$f(-2) = -1$

Endpoints

$f(-2) = f(2) - \int_{-2}^{2} f'(x)\,dx$

$= 1-(3-1)$

$= 1-2$

$= -1$

$A = \frac{\pi r^2}{2}$

$A_{sc} = \frac{4\pi}{2} = 2\pi$

$f(8) = f(2) + \int_{2}^{8} f'(x)\,dx$

$= 1 + (2 + (8-2\pi)) = 11-2\pi$

Question 4

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the graph of f', which consists of a semicircle and two line segments on the interval $-2 \le x \le 8$, is provided. It is given that f is defined on the closed interval $[-2, 8]$, and that $f(2) = 1$.

In part (a) students were asked to reason whether f has a relative minimum, relative maximum, or neither at $x = 6$. A correct response will use the given graph to reason that f' does not change signs at $x = 6$, although $f'(6) = 0$. Therefore f has neither a relative maximum nor a relative minimum at this point.

In part (b) students were asked to find all open intervals where f is concave down. A correct response will reason that a function is concave down when its first derivative is decreasing, and therefore f is concave down on the intervals $(-2, 0)$ and $(4, 6)$.

In part (c) students were asked to find $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$ or to show that this limit does not exist (and to justify their answer). A correct response will report that as x approaches 2, both the numerator and denominator of this ratio approach 0, and so L'Hospital's Rule applies. Using L'Hospital's Rule, the limit is shown to be 3.

In part (d) students were asked to find the absolute minimum value of f on the closed interval $[-2, 8]$ with a justification for their answer. A correct response will indicate the possible candidates for the location of the absolute minimum are the interval endpoints and the critical points $x = -1$, $x = 2$, and $x = 6$. A response could then reference work from part (a) to eliminate $x = 6$ as the location of a relative (or absolute) minimum, and could use the fact that the graph of f' changes from positive to negative at $x = -1$ in order to argue that a relative maximum occurs at $x = -1$. In addition, the given graph of f' indicates that $f'(x) \ge 0$ for $x > 2$, so the endpoint $x = 8$ cannot be the location of the absolute minimum value. The value of $f(2)$ is given in the stem of the problem, and using geometry, $f(-2) = 1 + \int_2^{-2} f'(x)\,dx = 3$. Therefore, the absolute minimum value of f on this closed interval is $f(2) = 1$. (Alternatively, a response could evaluate the function f at each of these five points and conclude that the absolute minimum is $f(2) = 1$.)

Sample: 4A
Score: 9

The response earned 9 points: 1 point in part (a), 2 points in part (b), 3 points in part (c), and 3 points in part (d).

In part (a) the response earned the point with a correct conclusion of no relative maximum or minimum and correct reasoning that there is no sign change for $f'(x)$ at $x = 6$.

In part (b) the response earned 2 points. The first point was earned with correct presentation of the intervals of concavity. The second point was earned with correct reasoning that $f'(x)$ is decreasing on these intervals.

In part (c) the response earned 3 points. The first point was earned with correct presentation of limits of the numerator and denominator. The second point was earned because the ratio of derivatives presented is correct. The third point was earned with a correct answer.

Question 4 (continued)

In part (d) the response earned 3 points. The first point was earned with the consideration of $f'(x) = 0.$ The response earned the second point with a correct analysis with a Candidates Test. The response earned the third point with a correct answer of 1.

Sample: 4B
Score: 6

The response earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d).

In part (a) the response earned the point with a correct conclusion of no relative maximum or minimum and correct reasoning that $f'(x)$ does not change signs at $x = 6.$

In part (b) the response earned 2 points. The first point was earned with correct presentation of the intervals of concavity. The second point was earned with correct reasoning that $f'(x)$ is decreasing.

In part (c) the response earned 2 points. The first point was not earned because the response does not present limits of the numerator and denominator. The second point was earned because the ratio of derivatives presented is correct. The third point was earned with a correct answer.

In part (d) the response earned 1 point. The first point was earned with the consideration of $f'(x)$ changing signs from negative to positive at $x = 2.$ The response did not earn the second point because there is no analysis with the endpoints or the elimination of interior points as possible minimums. The response is not eligible for the third point.

Sample: 4C
Score: 2

The response earned 2 points: no points in part (a), 1 point in part (b), no points in part (c), and 1 point in part (d).

In part (a) the response did not earn the point because the reasoning that there must be an inflection point at $f(6)$ is insufficient to earn the point.

In part (b) the response earned 1 point. The first point was earned with correct presentation of the intervals of concavity. The second point was not earned because the reasoning is based solely on $f''(x) < 0.$

In part (c) the response earned no points. The first point was not earned because the response does not present limits of the numerator and denominator. The second point was not earned because the ratio of derivatives presented is incorrect. The response is not eligible for the third point.

In part (d) the response earned 1 point. The first point was earned with the consideration of $f'(x) = 0$ in the first line. The response did not earn the second point because there is no analysis with the critical values $x = -1,$ $x = 2,$ and $x = 6.$ The response is ineligible for the third point.

2023

AP

AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 5

☑ Scoring Guidelines

☑ Student Samples

☑ Scoring Commentary

Part B (AB): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

The functions f and g are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of x.

	Model Solution	Scoring	
(a)	Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.		
	$h'(x) = f'(g(x)) \cdot g'(x)$ $h'(7) = f'(g(7)) \cdot g'(7)$	Chain rule	**1 point**
	$= f'(0) \cdot 8 = \frac{3}{2} \cdot 8 = 12$	Answer	**1 point**

Scoring notes:

- The first point is earned for either $h'(x) = f'(g(x)) \cdot g'(x)$ or $h'(7) = f'(g(7)) \cdot g'(7)$.
- If the first point is earned, the second point is earned only for an answer of 12 (or equivalent).
- If the first point is not earned, the second point can be earned only for a response of either $f'(0) \cdot 8 = 12$ or $\frac{3}{2} \cdot 8$.
- A response of 12 with no supporting work does not earn either point.

Total for part (a) — 2 points

	Model Solution	Scoring	
(b)	Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.		
	$k''(x) = 2f(x) \cdot f'(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$	Product or chain rule	**1 point**

$k''(4) = 2f(4) \cdot f'(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$	
$= 2 \cdot 4 \cdot 3 \cdot (-3) + 4^2 \cdot 2 = -72 + 32 = -40$	$k''(4)$ **1 point**
The graph of k is concave down at the point where $x = 4$ because $k''(4) < 0$ and k'' is continuous.	Answer with reason **1 point**

Scoring notes:

- The first point is earned for either $k''(x) = 2f(x) \cdot f'(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$ or $k''(4) = 2f(4) \cdot f'(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$.
- The first point is also earned by any of the following incorrect expressions, each of which has a single error in the application of the product rule or the chain rule:
  - $2f(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$ or $2f(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$
  - $2f'(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$ or $2f'(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$
  - $f'(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$ or $f'(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$
  - $2f(x) \cdot f'(x) \cdot g'(x)$ or $2f(4) \cdot f'(4) \cdot g'(4)$
  - Note: A response that presents one of these expressions cannot earn the second point.
- To earn the second point a response must correctly find $k''(4) = -40$ (or equivalent) with supporting work.
- The third point is earned for an answer and reason that are consistent with any declared nonzero value of $k''(4)$.

Total for part (b) 3 points

(c) Let m be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Find $m(2)$. Show the work that leads to your answer.

$m(2) = 5 \cdot 8 + \int_0^2 f'(t)\,dt = 40 + (f(2) - f(0))$ $= 40 + (7 - 10) = 37$	Answer with supporting work **1 point**

Scoring notes:

- The point is earned only for an answer of 37 (or equivalent) with supporting work equivalent to $5 \cdot 8 + (f(2) - f(0))$, $40 + (f(2) - f(0))$, $5 \cdot 8 + (7 - 10)$, or $40 + (7 - 10)$.
- An answer of 37 with no supporting work does not earn the point.

Total for part (c) 1 point

(d) Is the function m defined in part (c) increasing, decreasing, or neither at $x = 2$? Justify your answer.

$m'(x) = 15x^2 + f'(x)$	Considers $m'(x)$ **1 point**
$m'(2) = 15 \cdot 4 + f'(2) = 60 + (-8) = 52$	$m'(2)$ with supporting work **1 point**

The graph of m is increasing at $x = 2$ because $m'(2) > 0$.	Answer with justification	**1 point**

Scoring notes:

- The first point is earned for considering $m'(x)$, $m'(2)$, or m'. This consideration may appear in a justification statement.
- The second point is earned for $m'(2) = 15 \cdot 2^2 + f'(2)$, $m'(2) = 60 + f'(2)$, or $m'(2) = 60 - 8$ but is not earned for an unsupported response of $m'(2) = 52$.
- The third point is earned for an answer and justification consistent with any declared value of $m'(2)$.

Total for part (d) **3 points**

Total for question 5 **9 points**

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	−8	3	6
$g(x)$	1	2	−3	0
$g'(x)$	5	4	2	8

Response for question 5(a)

$h(x) = f(g(x)) \quad h'(x) = g'(x) \cdot f'(g(x))$

$h'(7) = g'(7) \cdot f'(g(7)) = 8 \cdot f'(0) = 8 \cdot \frac{3}{2}$

$= \frac{24}{2} = 12 \qquad h'(7) = 12$

Response for question 5(b)

$k'(x) = (f(x))^2 \cdot g(x)$

$k''(x) = 2f'(x)(f(x)) \cdot g(x) + g'(x)(f(x))^2$

$k''(4) = 2f'(4)f(4) \cdot g(4) + g'(4)(f(4))^2$

$= 2(3)(4)(-3) + 2(4)^2$

$= -72 + 32 = -40 \qquad -40 < 0$

The graph of k is concave down at $x = 4$ because $k''(x) < 0$.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0140676

Q5216/12

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

$m(x) = 5x^3 + \int_0^x f'(t)\,dt$

$m(2) = 5(2)^3 + \int_0^2 f'(t)\,dt = 5(8) + (f(2) - f(0))$

$= 40 + (7 - 10) = 40 - 3 = 37$

$m(2) = 37$

Response for question 5(d)

$m'(x) = 15x^2 + f'(x)$

$m'(2) = 15(2)^2 + f'(2) = 60 + (-8) = 52$

$52 > 0$

The function m is increasing at $x = 2$ because $m'(x) > 0$ at $x = 2$.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5216/13

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	−8	3	6
$g(x)$	1	2	−3	0
$g'(x)$	5	4	2	8

Response for question 5(a)

$h(x) = f(g(x))$

$h'(7) = f'(g(7)) \cdot (g'(7))$

$= f'(0) \cdot 8$

$= \frac{3}{2} \cdot 8$

$= \frac{24}{2} = \boxed{12}$

Response for question 5(b)

$2\cancel{3}2$
-24
7

$k'(4) = f(4)^2 \cdot g(4)$
$= 16 \cdot -3$

$k'(x) = (f(x))^2 \cdot g(x)$

$k''(4) = 2f(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$

$= (2(4) \cdot (-3)) + (16 \cdot 2)$

$= -24 + 32$

$= 7$

The graph is concave down becous the second derivative is pos but the first is neg

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

0045736

Q6171/12

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

$m(x) = 5x^3 + \int_0^x f'(t)\,dt$

$m(2) = 5(2)^3 + \int_0^2 f'(x)\,dx$

$= 40 + \int_0^2 f'(x)\,dx$

$f(x)\Big|_0^2$

$f(2) - f(0) = 7 - 10 = -3$

$40 + -3$

$m(2) = 37$

Response for question 5(d)

$m(0) = 5(0)^3 + \int_0^0 f'(x)\,dx$

The function defined in part C is increasing because the value @ m(2) is greater than @ m(0)

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 5 parts (a) and (b) on this page.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	−8	3	6
$g(x)$	1	2	−3	0
$g'(x)$	5	4	2	8

Response for question 5(a)

$h(x) = f(g(x))$ $h'(x) = f'(g(x)) + g'(x)$

$h'(7) = f'(g(7)) + g'(7)$

$h'(7) = \frac{3}{2} + 8$

$h'(7) = \frac{19}{2}$

Response for question 5(b)

$k'(x) = (f(x))^2 \cdot g(x)$

$k'(2) = (f(2))^2 \cdot g(2) = 94$

$k'(7) = (f(7))^2 \cdot g(7) = 0$

The graph of k is concave up at $x = 4$ because the graph of $k''(x)$ is negative, and the graph of $k'(x)$ starts out positive.

Answer QUESTION 5 parts (c) and (d) on this page.

Response for question 5(c)

$m(x) = 5x^3 + \int_0^x f'(t)\,dt$

$m(x) = 5x^3 + (f(x) - f(0))$

$m(2) = 5(2)^3 + (f(2) - f(0))$

$m(2) = 40 + (-3)$

$m(2) = 37$

Response for question 5(d)

$m'(x) = 15x^2 + f'(x) - f'(0)$

$m'(2) = 15(4) + f'(4) - f'(0)$

$m'(2) = \frac{123}{2}$

The function m is increasing at $x=2$ because $m'(2)$ is positive.

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem students were given a table of selected values of the twice-differentiable functions f and g and of their first derivatives.

In part (a) students are asked to find $h'(7)$ for the function $h(x) = f(g(x))$. A correct response will use the chain rule to find $h'(x) = f'(g(x)) \cdot g'(x)$, then pull the appropriate values from the given table to find $h'(7) = 12$.

In part (b) students were told that k is a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$ and were asked whether k is concave up or concave down at the point where $x = 4$. A correct response will use the product and chain rules to find $k''(x)$ and then evaluate $k''(4) = -40$ in order to determine that k is concave down at this point.

In part (c) the function $m(x) = 5x^3 + \int_0^x f'(t)\,dt$ is defined and students were asked to find $m(2)$. A correct response will use the Fundamental Theorem of Calculus to find $\int_0^2 f'(t)\,dt = f(2) - f(0)$, then use the given table to find $f(2)$ and $f(0)$. Finally, a correct response will combine the difference of these values with $5 \cdot 2^3$ to obtain $m(2) = 37$.

In part (d) students were asked whether this function m is increasing, decreasing, or neither at $x = 2$ and to provide a justification for their answer. A correct response will use the Fundamental Theorem of Calculus to find $m'(2) = 15 \cdot 2^2 + f'(2) = 52$ and realize that, because $m'(2)$ is positive, the function must be increasing in a neighborhood around $x = 2$.

Sample: 5A
Score: 9

The response earned 9 points: 2 points in part (a), 3 points in part (b), 1 point in part (c), and 3 points in part (d).

In part (a) the response earned the first point in line 1 on the right side with the correct chain rule. The numerical expression $8 \cdot \frac{3}{2}$ in line 2 on the right would have earned the second point with no simplification. In this case, correct simplification earned the point with $h'(7) = 12$.

In part (b) the response earned the first point in line 2 for the correct expression for $k''(x)$. The expression $2(3)(4)(-3) + 2(4)^2$ in line 4 would have earned the second point with no simplification. In this case, correct simplification to -40 in line 5 earned the point. The response earned the third point in line 6 for the correct answer and reason, "concave down at $x = 4$ because $k''(x) < 0$." This statement can be interpreted as $k''(4) < 0$ because $x = 4$ was stated in the stem of the question.

In part (c) the numerical expression $40 + (7 - 10)$ in line 3 would have earned the point with no simplification. In this case, correct simplification to 37 in line 3 earned the point.

Question 5 (continued)

In part (d) the response earned the first point in line 1 for considering $m'(x)$. The expression $15(2)^2 + f'(2)$ would have earned the second point with no simplification. In this case, correct simplification to 52 in line 2 earned the point. The response earned the third point in line 4 and line 5 for the correct conclusion with the correct reasoning.

Sample: 5B
Score: 4

The response earned 4 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response earned the first point in line 2 with the correct chain rule. The second point could have been earned for the numeric expression $\frac{3}{2} \cdot 8$ in line 4 but was simplified and the point was earned for the boxed answer of 12.

In part (b) the expression for $k''(4)$ in the form of a product rule with no evidence of a chain rule in line 2 earned the first point but is not eligible for the second point. The response did not earn the third point because the reason given "becaus the second derivative is pos but the first is neg" implies that concavity is based on both the first and second derivatives.

In part (c) the point could have been earned for the numeric expression $40 + -3$ on the right but was simplified and the point was earned for the boxed answer $m(2) = 37$ with supporting work.

In part (d) no points were earned because the response never considers $m'(x)$ or $m'(2)$.

Sample: 5C
Score: 3

The response earned 3 points: no points in part (a), no points in part (b), 1 point in part (c), and 2 points in part (d).

In part (a) the response did not earn the first point because the expression presented in line 1 on the right $h'(x) = f'(g(x)) + g'(x)$ is not the correct chain rule. The second point was not earned because the boxed answer $h'(7) = \frac{19}{2}$ is incorrect.

In part (b) the response earned no points because no expression for $k''(x)$ or $k''(4)$ is present.

In part (c) the point could have been earned for the numeric expression $40 + (-3)$ in line 4 but was simplified and the point was earned for the boxed answer $m(2) = 37$ with supporting work.

In part (d) the first point was earned in line 1 for considering $m'(x)$ even though the expression presented for $m'(x)$ is incorrect. The response did not earn the second point because the declared value $m'(2) = \frac{123}{2}$ is incorrect. The response earned the third point in lines 4 and 5 for the conclusion and justification consistent with the incorrect value declared for $m'(2)$.

2023

AP

AP® Calculus AB

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 6

- ☑ Scoring Guidelines
- ☑ Student Samples
- ☑ Scoring Commentary

Part B (AB): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the curve given by the equation $6xy = 2 + y^3$.

	Model Solution	Scoring	
(a)	Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$.		
	$\frac{d}{dx}(6xy) = \frac{d}{dx}\left(2 + y^3\right) \Rightarrow 6y + 6x\frac{dy}{dx} = 3y^2\frac{dy}{dx}$	Implicit differentiation	**1 point**
	$\Rightarrow 2y = \frac{dy}{dx}\left(y^2 - 2x\right) \Rightarrow \frac{dy}{dx} = \frac{2y}{y^2 - 2x}$	Verification	**1 point**

Scoring notes:

- The first point is earned only for the correct implicit differentiation of $6xy = 2 + y^3$. Responses may use alternative notations for $\frac{dy}{dx}$, such as y'.
- The second point cannot be earned without the first point.
- It is sufficient to present $2y = \frac{dy}{dx}\left(y^2 - 2x\right)$ to earn the second point, provided there are no subsequent errors.

Total for part (a) — 2 points

	Model Solution	Scoring	
(b)	Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal, or explain why no such point exists.		
	For the line tangent to the curve to be horizontal, it is necessary that $2y = 0$ (so $y = 0$) and that $y^2 - 2x \neq 0$.	Sets $2y = 0$	**1 point**
	Substituting $y = 0$ into $6xy = 2 + y^3$ yields the equation $6x \cdot 0 = 2$, which has no solution. Therefore, there is no point on the curve at which the line tangent to the curve is horizontal.	Answer with reason	**1 point**

Scoring notes:

- The first point is earned with any of $2y = 0$, $y = 0$, $\frac{dy}{dx} = 0$, $dy = 0$, $y' = 0$, or $\frac{2y}{y^2 - 2x} = 0$.
- A response need not state that at a horizontal tangent, $y^2 - 2x \neq 0$.

Total for part (b) 2 points

(c) Find the coordinates of a point on the curve at which the line tangent to the curve is vertical, or explain why no such point exists.

For a line tangent to this curve to be vertical, it is necessary that $2y \neq 0$ and that $y^2 - 2x = 0$ (so $x = \frac{y^2}{2}$).	Sets $y^2 - 2x = 0$	**1 point**
Substituting $x = \frac{y^2}{2}$ into $6xy = 2 + y^3$ yields the equation $3y^2 \cdot y = 2 + y^3 \Rightarrow 2y^3 = 2 \Rightarrow y = 1.$	Substitutes $x = \frac{y^2}{2}$ into $6xy = 2 + y^3$	**1 point**
Substituting $y = 1$ in $6xy = 2 + y^3$ yields $6x = 2 + 1$, or $x = \frac{1}{2}$. The tangent line to the curve is vertical at the point $\left(\frac{1}{2}, 1\right)$.	Answer	**1 point**

Scoring notes:

- The first point can be earned by presenting $y^2 = 2x$ or $y = \sqrt{2x}$.
- The second point can be earned for the substitution of $y = \sqrt{2x}$ into $6xy = 2 + y^3$, or for substituting $x = \frac{2 + y^3}{6y}$ into $y^2 - 2x = 0$.
- A response earns all three points by setting $y^2 - 2x = 0$, declaring the point $\left(\frac{1}{2}, 1\right)$, and verifying that this point is on the curve $6xy = 2 + y^3$.
- A response that identifies the point $\left(\frac{1}{2}, 1\right)$ but does not verify that the point is on the curve, does not earn the second or the third point.
- To earn the third point the response must present both coordinates of the point $\left(\frac{1}{2}, 1\right)$. The coordinates need not appear as an ordered pair as long as they are labeled.

Total for part (c) 3 points

(d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$ unit per second. What is the value of $\frac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

$6y\frac{dx}{dt} + 6x\frac{dy}{dt} = 0 + 3y^2\frac{dy}{dt}$	Uses implicit differentiation with respect to t	**1 point**
At the point $(x, y) = \left(\frac{1}{2}, -2\right)$, $6(-2)\left(\frac{2}{3}\right) + 6\left(\frac{1}{2}\right)\frac{dy}{dt} = 3(-2)^2\frac{dy}{dt}$ $\Rightarrow -8 + 3\frac{dy}{dt} = 12\frac{dy}{dt}$ $\Rightarrow \frac{dy}{dt} = -\frac{8}{9}$ unit per second	Answer	**1 point**

Scoring notes:

- The first point is earned by presenting one or more of the terms $6y\frac{dx}{dt}$, $6x\frac{dy}{dt}$, or $3y^2\frac{dy}{dt}$.
- Units will not affect scoring in this part.
- An unsupported response of $-\frac{8}{9}$ earns no points.
- Alternate solution:

  $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

  $$\left.\frac{dy}{dx}\right|_{(x,\, y)=(1/2,\, -2)} = \frac{2(-2)}{(-2)^2 - 2(1/2)} = -\frac{4}{3}$$

  $$\left.\frac{dy}{dt}\right|_{(x,\, y)=(1/2,\, -2)} = \left.\frac{dy}{dx} \cdot \frac{dx}{dt}\right|_{(x,\, y)=(1/2,\, -2)} = -\frac{4}{3} \cdot \frac{2}{3} = -\frac{8}{9} \text{ unit per second}$$

  - The first point is earned for the statement $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ or equivalent.
  - A numerical expression, such as $-\frac{4}{3} \cdot \frac{2}{3}$ or $\frac{2(-2)}{(-2)^2 - 2\left(\frac{1}{2}\right)} \cdot \frac{2}{3}$, earns both points.

Total for part (d) 2 points

Total for question 6 9 points

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Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$6xy = 2 + y^3$

$6y + 6x\frac{dy}{dx} = 3y^2\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{6y}{3y^2 - 6x}$

$\boxed{\frac{dy}{dx} = \frac{2y}{y^2 - 2x}}$

Response for question 6(b)

$\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$

horizontal tangent : $2y = 0$

$y = 0$

$6x(0) = 2 + 0$

$0 \neq 2$

therefore, since there are no x-values at y=0, there are no ~~points~~ points with a horizontal tangent

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$\frac{dy}{dx} = \frac{2y}{y^2-2x}$

vertical tangent: $y^2-2x=0$

$x = \frac{y^2}{2}$

$6(\frac{y^2}{2})(y) = 2+y^3$

$3y^3 = 2+y^3$

$2y^3 = 2$

$y^3 = 1 \rightarrow y=1, \; x=\frac{1}{2}$

at $(\frac{1}{2}, 1)$, there is a vertical tangent

Response for question 6(d)

$6xy = 2+y^3$

$6y\frac{dx}{dt} + 6x\frac{dy}{dt} = 3y^2\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{6y\, dx/dt}{3y^2-6x}$

$\frac{dy}{dt} = \frac{6(-2)(2/3)}{3(-2)^2-6(1/2)} = \frac{-8}{12-3}$

$\frac{dy}{dt} = -\frac{8}{9}$ units per second

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$y^3 + 2 = 6xy$

$3y^2 \frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$

$3y^2 \frac{dy}{dx} - 6x\frac{dy}{dx} = 6y$

$\frac{dy}{dx}(3y^2 - 6x) = 6y$

$\frac{dy}{dx} = \frac{6y}{3y^2 - 6x} = \frac{2y}{y^2 - 2x}$

Response for question 6(b)

$0 = \frac{2y}{y^2 - 2x}$

$y = 0$

$\frac{0}{0 - 2x} = \frac{0}{0 - 2x}$

$(0, 0)$

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

DNE since vertical tangent cannot exist on a function

Response for question 6(d)

$\left(\frac{1}{2}, -2\right)$

$2+y^3 = 6xy$

$2 + 3y^2 \frac{dy}{dt} = 6\frac{dx}{dt}y + 6x\frac{dy}{dt}$

$2 + 12\frac{dy}{dt} = -6 + 3\frac{dy}{dt}$

$12\frac{dy}{dt} - 3\frac{dy}{dt} = 10$

$9\frac{dy}{dt} = 10$

$\frac{2}{3}$ 6 4 8 +

$\frac{dy}{dt} = \frac{10}{9}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$6xy = 2 + y^3$

$6\left(x\frac{dy}{dx}\ y\frac{dx}{dy}\right) = 3y^2\frac{dy}{dx}$

$x\frac{dy}{dx}\ y\frac{dx}{dy} = \frac{1}{2}.y^2\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2y}{y^2-2x}$

Response for question 6(b)

$2y = 0$

$y = 0$

$(0,0)$

$y^2 - 2x = 0$

$y^2 = 2x$

$y = \sqrt{2x}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$y^2 - 2x = 0$

$y^2 = 2x$

$y = \sqrt{2x}$

$(0, \sqrt{2x})$

Response for question 6(d)

$\frac{dy}{dt} = \frac{2y}{y^2 - 2x}$

$\frac{dy}{dt} = \frac{2(-2)}{(-2)^2 - 2(1/2)} = \frac{-4}{4-1} = \boxed{-\frac{4}{5}}$

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

This problem asked students to consider the curve defined by the equation $6xy = 2 + y^3$.

In part (a) students were asked show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$. A correct response will implicitly differentiate the equation $6xy = 2 + y^3$ with respect to x, then solve the resulting equation for $\frac{dy}{dx}$.

In part (b) students were asked to find the coordinates of a point on the curve at which the tangent line is *horizontal*, or to explain why no such point exists. A correct response will note that a horizontal tangent line must have $\frac{dy}{dx} = 0$, which requires $2y = 0$ and, therefore, $y = 0$. But if $y = 0$, using the given equation $6xy = 2 + y^3$ yields $6x \cdot 0 = 2$, which has no solution. Therefore, there is no point on this curve at which the tangent line is horizontal.

In part (c) students were asked to find the coordinates of a point on the curve at which the tangent line is *vertical*, or to explain why no such point exists. A correct response will begin by noting that such a point requires $y^2 - 2x = 0 \Rightarrow x = \frac{y^2}{2}$. Substituting into the equation $6xy = 2 + y^3$ yields $y = 1$ and then $x = \frac{1}{2}$, resulting in a vertical tangent line at the point $\left(\frac{1}{2}, 1\right)$.

In part (d) students were asked to find the value of $\frac{dy}{dt}$ at the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, given that at that instant the particle's horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$. A correct response will implicitly differentiate the equation $6xy = 2 + y^3$ with respect to t and then solve the resulting equation for $\frac{dy}{dt}$ using $x = \frac{1}{2}$, $y = -2$, and $\frac{dx}{dt} = \frac{2}{3}$.

Sample: 6A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 3 points in part (c), and 2 points in part (d).

In part (a) the response earned the first point on the second line with the equation $6y + 6x\frac{dy}{dx} = 3y^2\frac{dy}{dx}$, the correct implicit differentiation for the given curve. The response correctly solves for $\frac{dy}{dx}$ on the third line and then earned the second point on the last line with the boxed equation.

Question 6 (continued)

In part (b) the response earned the first point on the second line with the equation $2y = 0$. The response earned the second point with the correct answer that "there are no points with a horizontal tangent," together with reason that "there are no x -values at $y = 0$."

In part (c) the response earned the first point on the second line with the equation $y^2 - 2x = 0$. The response then earned the second point on the fourth line with the correct substitution. The response would have earned the third point on the seventh line with the statements $y = 1$ and $x = \frac{1}{2}$; however, the response restates the answer correctly as an ordered pair and earned the point with the boxed section on the last line.

In part (d) the response earned the first point on the second line with the correct implicit differentiation of the curve with respect to t. The response would have earned the second point with the middle expression on the fourth line; however, the response presents two correct simplifications of this numerical answer and earned the point with the boxed answer.

Sample: 6B
Score: 4

The response earned 4 points: 2 points in part (a), 1 point in part (b), no points in part (c), and 1 point in part (d).

In part (a) the response earned the first point with the equation on the second line. The response correctly solves for $\frac{dy}{dx}$ on the subsequent three lines and earned the second point on the last line with the final simplification.

In part (b) the response earned the first point on the first line with the equation $0 = \frac{2y}{y^2 - 2x}$. The response then concludes that $y = 0$, which would also have earned the first point. The response did not earn the second point as there is no conclusion stating that no point exists.

In part (c) the response did not earn the first point as there is no evidence that the denominator of our presented $\frac{dy}{dx}$ has been set equal to 0. The response does not indicate any substitution, so it did not earn the second point. Finally, as the correct point is not presented, the response did not earn the third point.

In part (d) the response states that $2 + 3y^2 \frac{dy}{dy} = 6\frac{dx}{dt}y + 6x\frac{dy}{dt}$ on the third line. While this is not the correct implicit differentiation of the given curve with respect to t, because at least one of the three terms involving the rates $\frac{dx}{dt}$ or $\frac{dy}{dt}$ is correct, the response earned the first point on this line. The response did not earn the second point because the answer presented is not correct.

Sample: 6C
Score: 2

The response earned 2 points: no points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response did not earn the first point as no correct implicit differentiation of the given curve is presented, except for the one that was given in the stem of the problem. Because the first point was not earned, the response did not earn the second point.

Question 6 (continued)

In part (b) the response earned the first point with the equation $2y = 0.$ Note that the next equation $y = 0$ would also have earned this point. As the response never concludes that no such point exists, the response did not earn the second point.

In part (c) the response earned the first point on the first line with the equation $y^2 - 2x = 0.$ Note that any of the first three lines would have earned this first point. As there is no substitution presented and the correct answer is not stated, the response earned neither the second point nor the third point.

In part (d) the response states on the first line that $\frac{dy}{dt} = \frac{2y}{y^2 - 2x}$ and then uses this expression as the basis for substitution with the given values for x and $y.$ If this expression is viewed as the implicit differentiation of the curve with respect to $t,$ then the response does not present at least one correct term with a rate $\frac{dx}{dt}$ or $\frac{dy}{dt}.$ If, on the other hand, this expression is meant to be $\frac{dy}{dx},$ then the solution presented never makes use of $\frac{dx}{dt}.$ In either case, the response did not earn the first point. As the stated answer is incorrect, the response did not earn the second point.